Quantum Causality

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VI Quantum Information School, Paraty 2017, 14th – 18th August
Order of events is determined by their position in space-time

\[ A < B \]

\[ A \parallel B \]

\[ B < A \]

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \]

\[ ds^2 > 0 \quad \text{space-like sep.} \]

\[ ds^2 = 0 \quad \text{light-like sep.} \]

\[ ds^2 < 0 \quad \text{time-like sep.} \]
1957 Chapel Hill Conference, Richard Feynman

\[ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \simeq \frac{1}{\sqrt{2}}(|g^{0}_{\mu\nu}\rangle + |g^{1}_{\mu\nu}\rangle) \]

GR: dynamical structure of space-time + QT: superposition principle
1957 Chapel Hill Conference, Richard Feynman

\[
\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \simeq \frac{1}{\sqrt{2}}(|g^0_{\mu\nu}\rangle + |g^1_{\mu\nu}\rangle)
\]

\(g_{\mu\nu}\) is not well-defined and hence not \(ds^2\)

GR: dynamical structure of space-time + QT: superposition principle
Motivation

- Can one formulate quantum theory without the assumption of background space-time or predefined causal structure?

- Can such a formulation describe correlations for which the causal ordering of events is quantum mechanically indefinite?

- Can a “quantum gravity computer” have a greater computational power than standard (causal) quantum circuits?

L. Hardy, gr-qc/0509120, gr-qc/0608043, quant-ph/0701019
Outline

- Device-independent approach to causality
  Causal inequalities

- Framework for quantum mechanics with no global causal structure:
  Causally non-separable processes
  The quantum switch

- Quantum computation and communication on indefinite causal structures

- Physical realization of causally non-separable processes via superposition of large masses
Device-independent approach to causality: Causal inequalities and causal polytope
“Correlation does not imply causation”

Correlations: Whenever the sun rises, the rooster crows.

Crowing rooster causes the sun to rise?  
Rising sun causes the rooster to crow?

Need for interventions ("free variables") independent of the two:

a: The sun is rising or not  
b: The rooster is crowing or not  
x: Switching the sun on & off (hard)  
y: Making a chicken soup or not

\[
\sum_{a} p(a, b|x, y) = p(b|x, y) \\
\sum_{b} p(a, b|x, y) = p(a|x)
\]

Conclusion: The sun will rise even if we cook the soup, but the rooster will not crow, if we switch off the sun.
Free variables are variables that are statistically independent of "the rest of the experiment". Other variables may depend on the free variables.

In space-time $x_i$ is a "free variable" if it is independent of anything outside of its future lightcone

$$p(x_i | x_1, \ldots, x_{i-1}) = p(x_i)$$


Free variables are variables that are statistically independent of "the rest of the experiment".

Other variables may depend on the free variables.

"non-free variable": the last domino falls down or not

$$p(a | x)$$

"free variable": coin toss decides whether to push or not the first domino
Definite causal order

One-directional signalling
“from the past to the future”

Causal polytopes

Positivity: \( p^{A \leq B}(a, b|x, y) \geq 0 \quad \forall a, b, x, y \)

Normalisation: \( \sum_{a, b} p^{A \leq B}(a, b|x, y) = 1 \quad \forall x, y \)

No-signaling from B to A: \( \sum_{b} p^{A \leq B}(a|x, y) = \sum_{b} p^{A \leq B}(a|x, y') \quad \forall a, x, y, y' \)

Linear constraints on a bounded probability space

The probability space: the set of points
\[
\begin{pmatrix}
p^{A \leq B}(a, b|x, y) \\
\cdots \\
\cdots
\end{pmatrix}
\]
in a high dimensional space

Causal polytopes

Positivity: \( p^{A \leq B}(a, b|x, y) \geq 0 \quad \forall a, b, x, y \)

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Linear constraints on a bounded probability space

Causal polytopes

Positivity: \( p_{A \leq B}(a, b|x, y) \geq 0 \quad \forall a, b, x, y \)

Normalisation: \( \sum_{a, b} p_{A \leq B}(a, b|x, y) = 1 \quad \forall x, y \)

No-signaling from B to A: \( \sum_{b} p_{A \leq B}(a| x, y) = \sum_{b} p_{A \leq B}(a| x, y') \quad \forall a, x, y, y' \)

**Linear constraints** on a bounded probability space

Polytope \( A \leq B \)

Number of vertices \( k_A^m_A k_B^m_B \)

Number of functions \( \alpha(x) \) from \( k_A \) inputs to \( m_A \) outputs

Number of functions \( \beta(x, y) \) from \( k_B \) inputs to \( m_B \) outputs

Causal inequalities

Causal correlations: either A signals to B or B signals to A, or no-signalling or a convex combination of these situations

\[ p^{caus}(a, b|x, y) = \lambda p^{A \leq B}(a, b|x, y) + (1 - \lambda) p^{B \leq A}(a, b|x, y) \]

(Tight) causal inequalities

Causal correlations satisfy causal inequalities, which are facets of the causal polytope.

Number of vertices:

\[ k^m_A k^m_B - k^m_A k^m_B \]

The simplest causal inequality

Causal inequalities for 2 inputs, 2 outputs:

112 deterministic vertices, 48 facets, 16 of these facets are trivial, after relabelling 2 non-trivial facets remain:

“Guess my neighbour’s input game” (GYNI)

\[
p(a = y, b = x) \equiv \frac{1}{4} \sum_{x,y,a,b} \delta_{x,b} \delta_{y,a} \ p(a, b|x, y) \leq \frac{1}{2}
\]

“Lazy” version of the GYNI game

\[
p(x(a \oplus y) = 0, y(b \oplus x) = 0) \equiv \frac{1}{4} \sum_{x,y,a,b} \delta_{x(a \oplus y),0} \delta_{y(b \oplus x),0} \ p(a, b|x, y) \leq \frac{1}{2}
\]

"Non-causal correlations"

\[ p^{non-caus}(a, b|x, y) \neq \lambda p^{A\leq B}(a, b|x, y) + (1 - \lambda) p^{B\leq A}(a, b|x, y) \]

\[ p(a = y, b = x) > \frac{1}{2} \]

Interpretation: Both A signals to B and B signals to A, although the system enters only once the laboratory and the laboratories are shielded.

Causal correlations for many parties

\[ p(\vec{a} | \vec{x}) \equiv p(a_1, \ldots, a_N | x_1, \ldots, x_N) \]

Reduced probabilities:

\[ p(a_1, \ldots, a_i | \vec{x}) = \sum_{a_{i+1}, \ldots, a_N} p(\vec{a}, \vec{x}) \]

No-signaling from the last N-i parties:

\[ p(a_1, \ldots, a_i | x_1, \ldots, x_i, x_{i+1}, \ldots x_N) = p(a_1, \ldots, a_i | x_1, \ldots, x_i, x'_{i+1}, \ldots, x'_N) \quad (\star) \]

The probability \( p^{\text{order}}(\vec{a} | \vec{x}) \) is compatible with a definite causal order if (\star) is valid for all \( i \)

\[ A^1 \leq A^2 \leq \ldots \leq A^N \]

The probability is called causal if

\[ p(\vec{a} | \vec{x}) = \sum_{\mu} q_{\mu} p^{\text{order}}(\vec{a} | \vec{x}) \quad \sum_{\mu} q_{\mu} = 1, q_{\mu} \geq 0 \]
Device-dependent approach to causality: Causally non-separable processes
Motivation

- Can one formulate quantum theory without the assumption of background space-time or predefined causal structure?

- Can such a formulation describe correlations for which the causal ordering of events is quantum mechanically indefinite?

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L. Hardy, gr-qc/0509120, gr-qc/0608043, quant-ph/0701019
Problem

In standard formulation of quantum theory, time-like and space-like separated scenarios are mathematically described in very different ways.

Time-like separation

$$\text{Prob} = \frac{\text{Tr}(\hat{P}_2 e^{-i\hat{H}(t_2-t_1)} \hat{P}_1 e^{-i\hat{H}t_1} \hat{\rho} e^{i\hat{H}t_1} \hat{P}_1 e^{i\hat{H}(t_2-t_1)} \hat{P}_2)}{\text{Tr}(e^{-i\hat{H}t_1} \hat{\rho} e^{i\hat{H}t_1} \hat{P}_1)}$$

Space-like separation

$$\text{Prob} = \text{Tr}(\hat{\rho} (\hat{P}_1 \otimes \hat{P}_2))$$
Intuitive picture

Locally causal, globally indefinite

“causal” region

“non-causal region"
Intuitive picture

Globally causal
Local quantum laboratory

Output Hilbert space $\mathcal{H}^2$

Input Hilbert space $\mathcal{H}^1$

Quantum (CP) map
General quantum correlations

Probabilities: \( p(M_a, M_b, M_c, ...) = ? \)
Local quantum laboratory

\begin{itemize}
  \item "Event" = completely positive (CP) trace-nonincreasing maps
    \( \mathcal{M}_a : \mathcal{L}(\mathcal{H}^1) \to \mathcal{L}(\mathcal{H}^2), \quad \{\mathcal{M}_a\}_{a=1}^d \)
  \item Choi-Jamiołkowski representation:
    \( M_a = [(\mathbf{1} \otimes \mathcal{M}_a)(|\Phi^+\rangle\langle\Phi^+|)]^T \in \mathcal{L}(\mathcal{H}^1) \otimes \mathcal{L}(\mathcal{H}^2), \quad M_a \geq 0 \)
  \item Completely positive trace preserving (CPTP) map
    \( M = \sum_a M_a, \quad \text{Tr}_2 M = \mathbf{1}^1 \)
\end{itemize}

In local lab (causal)
QM is valid.

\( |\Phi^+\rangle = \sum_i |i\rangle \otimes |i\rangle \)
Examples of maps

Choi-Jamiołkowski isomorphism

\[ M_a = \left[ (1 \otimes M_a)(|\Phi^+\rangle\langle\Phi^+|) \right]^T \]

Inversed Choi-Jamiołkowski isomorphism

\[ M_a(\rho) = \left[ \text{Tr}_1((\rho \otimes 1)M_a) \right]^T \]

Measurement and re-preparation

\[ M_a = E_a \otimes \sigma^T \]

Replacing the state

\[ M_a = 1 \otimes \sigma^T \]

POVM (here \( d_2 = 1 \))

\[ M_a = E_a \]

Unitary

\[ (1 \otimes U^*)|\Phi^+\rangle \]

\[ U = \sum_{j,k} u_{jk}|j\rangle\langle k| \]

Special case, Identity channel:

\[ |\Phi^+\rangle \equiv |1\rangle \]
Nonnegative probabilities: \( W^{A_1 A_2 B_1 B_2} \geq 0 \)

Probabilities sum to 1: 
\[
\text{Tr} \left[ W^{A_1 A_2 B_1 B_2} \left( M^{A_1 A_2} \otimes M^{B_1 B_2} \right) \right] = 1, \\
\forall M^{A_1 A_2}, M^{B_1 B_2} \geq 0, \quad \text{Tr}_{A_2} M^{A_1 A_2} = \mathbb{1}^{A_1}, \text{Tr}_{B_2} M^{B_1 B_2} = \mathbb{1}^{B_1}
\]
The parties may share ancillary entangled states which do not fix causal order. Probabilities should be positive even when the parties share entanglement.

\[ p(a, b) = \text{Tr}\left[ W^{A_1 A_2 B_1 B_2} \left( M_{a A_1 A_2} \otimes M_{b B_1 B_2} \right) \right] \]

Positive on pure tensors (POPT)

The parties may share ancillary entangled states which do not fix causal order. Probabilities should be positive even when the parties share entanglement.

\[ W^{A_1 A_2 B_1 B_2} \geq 0 \]

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H. Barnum et al. (2001)
H. Barnum et al. (2005)
**Not admissible terms**

(producing paradoxes)

\[ W^{A_1 A_2 B_1 B_2} = \sum_{\mu_1, \mu_2, \mu_3, \mu_4} a_{\mu_1, \mu_2, \mu_3, \mu_4} \sigma^{A_1}_{\mu_1} \otimes \ldots \otimes \sigma^{B_2}_{\mu_4} \]

\[
\begin{align*}
\sigma^{A_1}_{\mu_1} & \otimes \mathbb{1}_{\text{rest}} \\
\sigma^{A_1}_{\mu_1} & \otimes \sigma^{A_2}_{\mu_2} \otimes \mathbb{1}_{\text{rest}} \quad \text{type } A_1 \\
\sigma^{A_1}_{\mu_1} & \otimes \sigma^{A_2}_{\mu_2} \otimes \mathbb{1}_{\text{rest}} \quad \text{type } A_1, A_2
\end{align*}
\]

\[
\ldots
\]

<table>
<thead>
<tr>
<th>A_2, B_2, A_2 B_2</th>
<th>A_1 A_2, B_1 B_2</th>
<th>A_1 A_2 B_2, A_2 B_1 B_2</th>
<th>A_1 A_2 B_1 B_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Postselection</td>
<td>Local loops</td>
<td>Channels with local loops</td>
<td>Global loops</td>
</tr>
</tbody>
</table>

![Diagram](image-url)
Admissible terms

<table>
<thead>
<tr>
<th>$B \not\equiv A$</th>
<th>$A_1, B_1, A_1 B_1$</th>
<th>$A_2 B_1$</th>
<th>$A_1 A_2 B_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \not\equiv B$</td>
<td>$A_1 B_2$</td>
<td>$A_1 B_1 B_2$</td>
<td></td>
</tr>
<tr>
<td>Causal order</td>
<td>States</td>
<td>Channels</td>
<td>Channels with memory</td>
</tr>
</tbody>
</table>

- **States**
  - $A_1, B_1, A_1 B_1$
  - $A_2 B_1$
  - $A_1 A_2 B_1$

- **Channels**
  - $A_1 B_2$
  - $A_1 B_1 B_2$

- **Channels with memory**
  - $A_2$ connected to $B_1$
Causally ordered process matrices

\[ W = \rho^{A_1} \otimes C^{A_2 B_1} \otimes 1^{B_2} \]

Channel from A to B
Time-like separation

Channel with memory from A to B
Time-like separation

States
Space-like separation

\[ p(a, b) = \text{Tr}(\mathcal{M}_b \circ \mathcal{C} \circ \mathcal{M}_a(\rho)) \]

Inversed Choi-Jamilkowski isomorphism

\[ p(a, b) = \text{Tr}((E_a \otimes E_b)\rho) \]

Causally ordered process matrices

\[ W = \rho^{A_1} \otimes C^{A_2B_1} \otimes 1^{B_2} \]

\[ W = C^{A_1A_2B_1} \otimes 1^{B_2} \]

\[ W = \rho^{A_1B_1} \otimes 1^{A_2B_2} \]

Channel from A to B
Time-like separation

Channel with memory
from A to B
Time-like separation

States
Space-like separation

Process matrix formalism is a **unified quantum framework** to describe space-like and time-like separated scenarios.

Causally separable processes

Most general processes compatible with definite causal structure (convex mixtures of ordered processes):

\[ W = \lambda W^{A \leq B} + (1 - \lambda) W^{B \leq A} \]


**Multipartite generalization:** O. Oreshkov and C. Giarmatzi, New J. Phys. 18 093020 (2016)
Causally non-separable processes
(device-dependent notion of causality)

**Theorem:** There are processes (called **causally non-separable**) for which

\[ W \neq \lambda W^{A \leq B} + (1 - \lambda) W^{B \leq A} \]

**Example:** The quantum switch

\[ |W\rangle = |0\rangle^{cnt} |\psi\rangle^{A_1 |1\rangle^{A_2 B_1} |1\rangle^{B_2 C_1} + |1\rangle^{cnt} |\psi\rangle^{B_1 |1\rangle^{B_2 A_1} |1\rangle^{A_2 C_1} \]

Identity channel: \( |1\rangle\rangle = \sum_j |j\rangle |j\rangle \)


The “quantum switch”

\[ |W\rangle = |0\rangle_{cnt} |\psi\rangle^{A_1} |1\rangle^{A_2B_1} |1\rangle^{B_2C_1} + |1\rangle_{cnt} |\psi\rangle^{B_1} |1\rangle^{B_2A_1} |1\rangle^{A_1C_1} \]

**Theorem:** Quantum switch is a causally nonseparable process

**Sketch of the proof:**
- Process is extremal (pure)
- Only extremal causally separable processes are channels and states
- Quantum switch is neither of the two as it allows two-way signalling (contains both $W^{A_1A_2B_1}$ and $W^{B_1B_2A_1}$ terms)
- C is the last

\[ W \neq \lambda W^{A \preceq B \preceq C} + (1 - \lambda) W^{B \preceq A \preceq C} \]

Realization of the quantum switch

Preparation:
\[ |0\rangle_{cnt} + |1\rangle_{cnt} |\psi\rangle \]

Path degree of freedom (quantum control)

\[ |0\rangle_{cnt} U_A U_B |\psi\rangle + |1\rangle_{cnt} U_B U_A |\psi\rangle \]

Internal degree of freedom (polarization)

„Tracing out“ the quantum control results in a causally separable process
Experimental demonstration

The quantum switch has a causal model

- C is in future of A and B
- After tracing out C, A and B are causally separable

\[
p(a, b, c|x, y, z) = p(a, b|x, y, z)p(c|a, b, x, y, z)
\]

\[
= p(a, b|x, y)p(c|a, b, x, y, z)
\]

\[
= (\lambda p^{A \leq B}(a, b|x, y) + (1 - \lambda)p^{B \leq A}(a, b|x, y))p(c|a, b, x, y, z)
\]

\[
= \lambda p^{A \leq B \leq C}(a, b, c|x, y, z) + (1 - \lambda)p^{B \leq A \leq C}(a, b, c|x, y, z)
\]

Process matrices violate causal inequalities

Process:

\[
W = \frac{1}{4} \left[ 1 \otimes^4 + \frac{Z^{A_1} Z^{A_0} Z^{B_i} I^{B_0} + Z^{A_1} I^{A_0} X^{B_i} X^{B_0}}{\sqrt{2}} \right]
\]

\[
M^{A_1 A_0}_{0|0} = M^{B_i B_0}_{0|0} = 0,
\]

\[
M^{A_1 A_0}_{1|0} = M^{B_i B_0}_{1|0} = 2 \left| \Phi^+ \right\rangle \left\langle \Phi^+ \right|,
\]

\[
M^{A_1 A_0}_{0|1} = M^{B_i B_0}_{0|1} = \left| 0 \right\rangle \left\langle 0 \right| \otimes \left| 0 \right\rangle \left\langle 0 \right|,
\]

\[
M^{A_1 A_0}_{1|1} = M^{B_i B_0}_{1|1} = \left| 1 \right\rangle \left\langle 1 \right| \otimes \left| 0 \right\rangle \left\langle 0 \right|
\]

Local maps:

Identity channel

Measurement

Repreparation

\[
P_{GYNI} = \frac{5}{16} \left( 1 + \frac{1}{\sqrt{2}} \right) \approx 0.5335 > \frac{1}{2},
\]

\[
P_{LGYNI} = \frac{5}{16} \left( 1 + \frac{1}{\sqrt{2}} \right) + \frac{1}{4} \approx 0.7835 > \frac{3}{4}
\]

Can processes that violate causal inequalities be realized in nature?

Are such processes in contradiction with some well accepted physical principles?

\[ W = \frac{1}{4} \left[ 1 \otimes 4 + \frac{Z_{A} Z_{B} 1_{B} 1_{B} + Z_{A} Z_{A} X_{B} X_{B}}{\sqrt{2}} \right] \]
Quantum information processing on indefinite causal structures
Motivation

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L. Hardy, gr-qc/0509120, gr-qc/0608043, quant-ph/0701019
1. wires are quantum systems
2. each box is a single use of transformation
3. input-output proceeds from bottom to top and there are no loops
Quantum Circuit

Causal evolution in space-time
Quantum Circuit Process Matrix

"No past, no future"

A computational problem

Given two unitaries $U_0$ and $U_1$, decide whether $[U_0, U_1] = 0$ or $\{U_0, U_1\} = 0$, given the promise that one of them is realized.

The quantum switch allows to superpose ordering of the gates

Hadamard: $|C\rangle|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\psi\rangle$

2-switch: $S_2|C\rangle|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle U_1 U_0 |\psi\rangle + |1\rangle U_0 U_1 |\psi\rangle)$

Hadamard: $HS_2|C\rangle|\psi\rangle = \frac{1}{2}(|0\rangle \{U_0, U_1\} |\psi\rangle + |1\rangle [U_1, U_0] |\psi\rangle)$

A computational problem

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A computational problem

Given two unitaries $U_0$ and $U_1$, decide whether $[U_0, U_1] = 0$ or $\{U_0, U_1\} = 0$, given the promise that one of them is realized.

The best quantum circuit we know needs two queries (calls) of at least one of the boxes.

Experimental demonstration

Probability of success

Average measured: 0.970 ± 0.024
Max. causal: 0.92

Scaling the problem

Given a set of \( n \) unitaries \( U_0, \ldots, U_{n-1} \), decide for which \( y = 1, \ldots, n! \) it is true that

\[
\forall x \quad \prod_x = \omega^{xy} \prod_0, \quad \prod_x = \prod_{k=0}^{n-1} U_{x(k)}
\]

where \( \omega = e^{i2\pi/n!} \), \( x = 1, \ldots, n! \) given the promise that it is true for one \( y \).

Example for \( n = 2 \), the unitaries either commute or anticommute

The „n-switch“ solution

The n-switch: \( S_n |x\rangle |\psi\rangle = |x\rangle \Pi_x |\psi\rangle \)

Applies unitaries in the order of the permutation \( \Pi_x \)

Algorithm

Control in superposition: \(|C\rangle |\psi\rangle = \frac{1}{\sqrt{n!}} \sum_{x=0}^{n!-1} |x\rangle |\psi\rangle \)

The n-switch: \( S_n |C\rangle |\psi\rangle = \frac{1}{\sqrt{n!}} \sum_{x=0}^{n!-1} |x\rangle \Pi_x |\psi\rangle \)

The Fourier transform: \( \mathcal{F}_n S_n |C\rangle |\psi\rangle = \frac{1}{\sqrt{n!}} \sum_{x,y=0}^{n!-1} |y\rangle e^{-xy} \Pi_x |\psi\rangle \)

Measure the control: The result is \( y \)

The algorithm distinguishes between \( n! \) sets of \( n \) unitaries by querying each box only once (altogether \( n \) queries).

A solution with a fixed order of gates

Simulation of the n-switch (example for $n = 3$):

What is the length $m$ of the shortest sequence of $n$ characters containing all the permutations as subsequences?

Open problem in combinatorics, but the following bounds are known:

$$n^2 - C \epsilon n^{7/4 + \epsilon} \leq m \leq \left[ n^2 - \frac{7}{3} n + \frac{19}{3} \right]$$

A solution with a fixed order of gates

Simulation of the n-switch (example for \( n = 3 \)):

Reduction of the query complexity from \( O(n^2) \) to \( O(n) \)

Communication complexity problem

Communication complexity = number of (qu)bits transmitted

Find the protocol that minimizes the communication complexity

Exponential reduction in complexity: Deutsch-Jozsa, Raz’s problem
Consider binary distributed function \( f : X \times Y \rightarrow \{0, 1\} \) such that \( \forall x_1, x_2 \in X \) with \( x_1 \neq x_2 \), \( \exists y \in Y \) for which \( f(x_1, y) \neq f(x_2, y) \).

Then for deterministic evaluation the minimal dimension of the system that must be communicated between two parties sharing an arbitrary amount of entanglement is \( \sqrt{|X|} \).
Suppose that it is sufficient to send a system of dimension \( \sqrt{|X|} - 1 \). Then the maximal number of orthogonal states that can be transmitted by Alice to Bob is \((\sqrt{|X|} - 1)^2 < |X|\), by using superdense coding. Hence, there exists \( x_1, x_2 \in X \) such that their corresponding states are not orthogonal. But by our assumption, if Bob receives \( y \) for which \( f(x_1, y) \neq f(x_2, y) \), he would be able to perfectly distinguish them.
Exchange evaluation game

Alice's input:
\( x = x_1, \ldots, x_n \in \mathbb{Z}_2^n; \ f : \mathbb{Z}_2^n \to \mathbb{Z}_2 \)

Bob's input:
\( y = y_1, \ldots, y_n, \in \mathbb{Z}_2^n; \ g : \mathbb{Z}_2^n \to \mathbb{Z}_2 \)

Charlie needs to output
\[ EE_n(x, f, y, g) = f(y) \oplus g(x) \]

Dimension of the input \( 2^{2^n+n-1} \)

Exchange evaluation game has property \( \star \)

One-way quantum communication: \( \frac{1}{2}(2^n + n - 1) \) qubits
Superposition of directions of communication

Alice\'s encoding: \( U_A(x, f) = X(x)D(f) \) where: 
\[ X(x) = X^{x_1} \otimes \ldots \otimes X^{x_n} \]
\[ X^0 = 1, X^1 = \sigma_x \]
\[ D(f) = \sum_{z \in \mathbb{Z}_2^n} (-1)^{f(z)} |z\rangle \langle z| \]

Bob\'s encoding: \( U_B(y, g) = X(y)D(g) \) where: 
\[ X(y) = X^{y_1} \otimes \ldots \otimes X^{y_n} \]
\[ D(g) = \sum_{z \in \mathbb{Z}_2^n} (-1)^{g(z)} |z\rangle \langle z| \]

\[ [U_A(x, f), U_B(y, g)]|0\rangle = 0 \quad \text{if} \quad f(y) \oplus g(x) = 0 \]
\[ \{U_A(x, f), U_B(y, g)\}|0\rangle = 0 \quad \text{if} \quad f(y) \oplus g(x) = 1 \]

Superposition of directions: \( n \) qubits

Engineering Space-time: Creating causally non-separable processes
Experimental demonstration

The notion of an “event” in the quantum switch

Is application of a local unitary a single event?
And now for something completely different
Gravitational time-dilation

Initially synchronized clocks will eventually show different times when placed at different gravitational potentials.

Clock closer to a massive body ticks slower than the clock further away from the mass.
Gravitational time-dilation

Stationary metric, weak-field approximation:

\[ g_{00} \approx -\left(1 + 2 \frac{\Phi(x)}{c^2}\right) \]

\[ g_{rr} \approx \left(1 + 2 \frac{\Phi(x)}{c^2}\right)^{-1} \]

\[
\frac{\Delta \tau_L}{\Delta \tau_H} = 1 - \sqrt{\frac{g_{00}(H)}{g_{00}(L)}} = \\
= 1 - \frac{\Phi_H - \Phi_L}{c^2} = 1 - \frac{g \Delta x}{c^2}
\]

„lower clock is slower“
Order of events is determined by their position in space-time

\[ A \prec B \]

\[ A \parallel B \]

\[ B \prec A \]
General relativity: space-time is dynamical
General relativity: space-time is dynamical

Proper times at A and B

\[ \tau_A = \sqrt{\frac{g_{00}(r_A)}{g_{00}(r_B)}} \tau_B \]

Coordinate time of photon propagation between A & B

\[ T_c = \frac{1}{c} \int_{r_B}^{r_A} \sqrt{-\frac{g_{rr}(r')}{g_{00}(r')}} \, dr' \]

General relativity: space-time is dynamical

Position of the mass as a quantum control

Channel from B to A

$$\ket{1}^{cnt} \ket{\psi}^{B_1} \ket{1}^{B_2} \ket{A_1} \ket{1}^{A_2} \ket{C_1}$$

General relativity: space-time is dynamical

Position of the mass as a quantum control

Channel from A to B

\[ |0\rangle^{cnt} |\psi\rangle^{A_1} |\mathbb{1}\rangle^{A_2 B_1} |\mathbb{1}\rangle^{B_2 C_1} \]
Gravitational quantum switch

The quantum switch:

$$|W\rangle = |0\rangle^{cnt}|\psi \rangle^{A_1}|1\rangle^{A_2B_1}|1\rangle^{B_2C_1} + |1\rangle^{cnt}|\psi \rangle^{B_1}|1\rangle^{B_2A_1}|1\rangle^{A_2C_1}$$

Coordinate and proper time

Distant observer sees two enters in the lab at two coordinate times.

Local observers see a single enter at an instant of the proper time.

\[ \tau = \sqrt{-g_{00}(0)t_0} = \sqrt{-g_{00}(1)t_1} \]

„Earth left“  „Earth right“
Summary and outlook

- Global causal order need not be a necessary element of quantum theory.
- There exist causally nonseparable processes (the quantum switch).
- Linear advantage in computation and exponential reduction of communication complexity using the resource.
- There are processes that violate causal inequalities but we do not know whether they can be realized in nature (and whether they are describable by “standard” quantum theory).
- Not shown: “Entangled causal structures“ , „Bell‘s theorem for causal orders“, Causal witnesses etc.
- Future: We need more open minded (≈ young) people to work at the interplay between quantum gravity and quantum information.
Thank you!