EXERCISES FOR PARATY LECTURES (Richard Jozsa)

Exercise for Lecture 1

1D cluster state as an MPS

The 1D cluster state on a line on n qubits (labelled $1, 2, \ldots, n$) is the state $|Cl_n\rangle$ obtained as follows: start with a row of $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ states $|+\rangle_1 |+\rangle_2 \ldots |+\rangle_n$ and apply CZ to each nearest neighbour pair $(1,2), (2,3), \ldots, (n-1,n)$. Let $|i_1 \ldots i_n\rangle$ (with each i_k being 0 or 1) be any computational basis state.

(a) Show that $|Cl_n\rangle$ has amplitudes $\langle i_1 \ldots i_n | Cl_n\rangle = \frac{1}{2^n}(-1)^\alpha$ where α is the number of sites k with $i_k = i_{k+1} = 1$.

To express $|Cl_n\rangle$ as an MPS

$$|Cl_n\rangle = \sum_{i_1, i_2, \ldots, i_n = 0}^{1} L^{(i_n)} C^{(i_{n-1})} \ldots A^{(i_2)} R^{(i_1)} |i_1 \ldots i_n\rangle ,$$

where $A^{(i_2)}, \ldots, C^{(i_{n-1})}$ are matrices and $L^{(i_n)}$ and $R^{(i_1)}$ are row and column vectors respectively, we will use Dirac notation to write the amplitudes as $\left\langle L^{(i_n)} \right| C^{(i_{n-1})} \ldots A^{(i_2)} \left| R^{(i_1)} \right\rangle$. Introduce

$$T^{(0)} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = |p\rangle\langle 0| \qquad T^{(1)} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = |m\rangle\langle 1|$$

where $|p\rangle = |0\rangle + |1\rangle$ and $|m\rangle = |0\rangle - |1\rangle$.

(b) Show that $T^{(i)} T^{(i')} = \pm |p \text{ or } m\rangle\langle 0 \text{ or } 1|$ with a minus sign iff $i = i' = 1$.

What is the value of $\langle 0| T^{(i_1)} \ldots T^{(i_n)} |p\rangle$? Hence write down an MPS expression of $|Cl_n\rangle$ in the form $\sum_{i_1, i_2, \ldots, i_n} L^{(i_n)} C^{(i_{n-1})} \ldots A^{(i_2)} R^{(i_1)} |i_1 \ldots i_n\rangle$.

(c) Now consider any measurement-based quantum computational (MQC) process on the 1D cluster state. Such a process is defined by a sequence of 1-qubit measurements on specified qubits and each choice of measurement may depend on (classical poly-time) calculations with previous measurement outcomes. Can such a process be classically efficiently simulated? Give a justification of your answer by quoting suitable results from the lecture.

(d)**(optional extra)** It is known that the MQC process of (c) applied on a 2D cluster state is *universal* for quantum computing (this is standard Raussendorf-Briegel cluster state MQC) so we would not expect it to be efficiently classically simulatable! Why does the argument in (c) not apply for the 2D cluster state?

(Hints: Consider a square grid of $n \times n$ $|+\rangle$ states with CZ applied to each horizontal and vertical nearest neighbour pair in the grid. Suppose n is even for convenience. Let A be the subset of qubits in rows 1 to $n/2$ and B those in rows $n/2 + 1$ to n.

(i) Look at the qubits just in rows $n/2$ and $n/2 + 1$ (n qubits in each row). Show that the n vertical CZ operations on just these qubits gives a state with Schmidt rank 2^n.

(ii) Show that the Schmidt rank of any bipartite state of AB is unchanged by any unitary operations acting wholly within A or B.

(iii) Deduce that the Schmidt rank of the 2D cluster state is exponential in n for at least some partitions of the qubits, and conclude that our results about efficient classical simulaton do not apply to 2D cluster state MQC.

Exercise for Lecture 2

Clifford circuits

(a) Let C be a Clifford operation for n qudits. Then for any $T(\underline{a}, \underline{b}) = X^{a_1} Z^{b_1} \otimes \ldots \otimes X^{a_n} Z^{b_n}$ we have $C\, T(\underline{a}, \underline{b})\, C^\dagger = k\, T(\underline{a}', \underline{b}')$ for some $\underline{a}', \underline{b}' \in \mathbb{Z}_d^n$ and complex k. Show that k is necessarily a d^{th} root of unity. (Hint: recall $XZ = \omega ZX$ and $X^d = Z^d = I$).

(b) Consider Clifford circuits on n qubits with the following features: inputs are computational basis states; intermediate measurements are allowed within the circuit but with non-adaptive choice of later gates. We saw in the lecture that in this situation, single line outputs are classically strongly efficiently simulatable (e.g. by Thm 1 on slide 13, with its proof there too).
(i) Suppose the output now arises from measurements on $O(\log n)$ lines. By suitably generalising the given proof of Thm 1, show that this is still classically strongly efficiently simulatable. (Hint: recall $|0\rangle\langle 0| = (I + Z)/2$ and similarly for $|1\rangle\langle 1|$.)
(ii) Does this proof still hold if there are $O(n)$ output lines? Why (not)? (However this case is still classically strongly efficiently simulatable, but by a more complicated argument, see e.g. arXiv:1305.6190 theorem 4).

Exercise for Lecture 3

Jordan-Wigner matrices and matchgates

Let $c_1, c_2, \ldots, c_{2n}$ be the standard Jordan-Wigner (JW) matrices for n qubit lines. Consider the quadratic hamiltonian $H = i\theta c_1 c_2$ for real θ.

(a) Compute the associated matchgate (Gaussian operation) $U = \exp iH$.

(b) Calculate its conjugation action on the JW matrices and show directly that the linear span is preserved. Hence also find the corresponding rotation matrix $R \in SO(2n, \mathbb{R})$ associated to this conjugation action.

(c) **(optional extra)** (linear terms in the hamiltonian)
Now let $c_1, c_2, \ldots, c_{2n}$ be $2n$ abstract generators of a Clifford algebra (i.e. they satisfy the Clifford algebra anti-commutation relations). Introduce a further symbol called c_0 which is required to have the same relations with all the c_μ's i.e.

$$\{c_0, c_\mu\} = 2\delta_{0\mu} I \quad \text{for } \mu = 0, 1, \ldots, 2n$$

extending the set of c's to $2n + 1$ generators. Introduce

$$d_0 = c_0 \text{ and } \quad d_\mu = ic_\mu c_0 \text{ for } \mu = 1, \ldots, 2n.$$

(The optional factor i here is just to have all d's hermitian if the c's were.)
Show the following:
(i) $\{d_\mu, d_\nu\} = 2\delta_{\mu\nu} I \quad \mu, \nu = 0, 1, \ldots, 2n.$
(ii) A general purely quadratic expression in the d_μ's for $\mu = 0, 1, \ldots, 2n$

$$\tilde{A} = \sum_{\mu,\nu=0}^{2n} \tilde{a}_{\mu\nu} d_\mu d_\nu$$

is the same as a general quadratic *plus linear* expression in the original c_μ's for $\mu = 1, \ldots, 2n$

$$A = \sum_{\mu,\nu=1}^{2n} a_{\mu\nu}\, c_\mu c_\nu + \sum_{\sigma=1}^{2n} b_\sigma c_\sigma.$$

In fact $a_{\mu\nu} = \tilde{a}_{\mu\nu}$ for $\mu, \nu = 1, \ldots, 2n$ and $b_\sigma = i(\tilde{a}_{\sigma 0} - \tilde{a}_{0\sigma})$.

Hence Gaussian operations constructed from purely quadratic hamiltonians in the d_μ's correspond to an extended class of "generalised" Gaussian operations in the c_μ's viz. those arising from exponentials of *linear* as well as quadratic terms in the hamiltonian. Thus using our classical simulation methods applied to the d_μ's, arbitrary circuits of these generalised gates in the c_μ's can also be strongly classically efficiently simulated. See R.J, A. Miyake, S. Strelchuk arXiv:1311.3046v2 for more details and further developments of this.